Magnetic translation symmetry on the lattice

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ABSTRACT

Magnetic translation symmetry on a finite periodic square lattice is investigated for an arbitrary uniform magnetic field in arbitrary dimensions. It can be used to classify eigenvectors of the Hamiltonian. The system can be converted to another system of half or lower dimensions. A higher dimensional generalization of Harper equation is obtained for tight-binding systems.

1 Introduction

Systems of charged particles interacting with uniform magnetic fields have served as testing grounds of various ideas in quantum field theory and condensed matter physics. Uniform magnetic fields are simple and in many cases exact solutions are available while keeping some essential aspects of the interactions with the gauge fields. A remarkable feature of these system is the appearance of magnetic translation symmetry [1].

In the presence of magnetic fields the gauge potential depends on the coordinates. The change in the gauge potential due to a translation can be compensated by a suitable gauge transformation. Arbitrary translations become a symmetry of the Hamiltonian in infinite euclidean spaces and each eigenstate is infinitely degenerated. In finite periodic spaces or on compact tori the boundary conditions for the wave functions become twisted. Because of this only a discrete finite subgroup of the whole translations survives as magnetic translation symmetry. The degeneracy of states is a finte constant depending on the magnetic field.

On finite periodic lattices exact solutions are not available and translation symmetries are rather restricted. The spectra are far richer than those of the continuum [2]. Nevertheless, it is possible to define magnetic translations, which enable us to qualitatively understand the richness of the spectra. In this paper we investigate magnetic translation symmetry on the lattice in full detail for arbitrary uniform magnetic fields in arbitrary higher dimensions. The key idea is to introduce oblique lattice where the magnetic field takes a block-diagonal form composed of 2×2 anti-symmetric matrices. According to the rank of magnetic field, the generators of magnetic translations can be classified into pairs satisfying two dimensional magnetic translation algebra and unpaired commuting generators. This enables us to identify maximal commuting subset of the generators. Representation theoretic approach [3] can be used to reduce Hamiltonians into block-diagonal forms without relying on its concrete form. The reduced Hamiltonian describes a system of half or lower than half dimensions. This phenomenon has been observed for systems of electrons in periodic potentials [4]. Magnetic translation symmetry has been investigated in four dimensions in the context of chiral lattice fermions [5]. Reduction from two to one dimension and the spectra of hermitian Wilson-Dirac operator have also been discussed in Ref. [6].

This paper is organized as follows. In the next section we show that periodic boundary conditions on finite lattice can be converted to twisted periodicity on infinite lattice. In Sect. 3 we examine magnetic translations on the lattice. Constraints on the translation vectors are solved. To illustrate the approach we argue magnetic translation symmetry in two dimensions. This is done in Sect. 4. We introduce oblique lattice coordinates in Sect. 5 and give a generalization to arbitrary dimensions. Sect. 6 is devoted to summary and discussion.

2 Twisted boundary conditions

Let us consider a system of a charged particle interacting with a uniform magnetic field of an abelian gauge theory on a d dimensional periodic lattice of a size L^d, where L is a positive integer. We take the lattice constant $a = 1$. Any uniform magnetic field $F_{\mu\nu} = -F_{\nu\mu}$ ($\mu, \nu = 1, \cdots, d$) on the lattice can be written as

$$F_{\mu\nu} = \frac{2\pi m_{\mu\nu}}{L^2}, \tag{2.1}$$

where $m_{\mu\nu}$ are integers. These classify topological sectors of the lattice gauge fields [7, 8].

The link variables $U_\mu(x)$ giving rise to the uniform magnetic field can be chosen as [7]

$$U_\mu(x) = \exp\left[-\frac{2\pi i}{L}\delta_{\bar{x}_\mu, L-1}\sum_{\nu > \mu} m_{\mu\nu}\bar{x}_\nu - \frac{2\pi i}{L^2}\sum_{\nu < \mu} m_{\mu\nu}\bar{x}_\nu + ib_\mu\right], \tag{2.2}$$

where $0 \leq \bar{x}_\mu < L$ $(\mu = 1, \cdots, d)$ stand for periodic lattice coordinates satisfying $\overline{x_\mu + L} = \bar{x}_\mu$ and b_μ are real constants. These satisfy periodic boundary conditions

$$U_\mu(x + L\hat{\nu}) = U_\mu(x), \tag{2.3}$$

where $\hat{\nu}$ is the unit vector along the ν-th lattice axis. The magnetic fields (2.1) are related to the plaquette variables by

$$U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x) = e^{iF_{\mu\nu}}. \tag{2.4}$$

We denote the Hamiltonian or the lattice Dirac operator of the system by H and consider the eigenvalue problem

$$H\psi(x) = \lambda\psi(x). \tag{2.5}$$

Concrete form of H is not necessary to argue magnetic translation symmetries. We only assume that H only depends on the lattice coordinates through the link variables. The wave function $\psi(x)$ is subject to the the periodic boundary conditions

$$\psi(x + L\hat{\mu}) = \psi(x). \tag{2.6}$$

In lattice gauge theory it is customary to adopt periodic boundary conditions. The lattice is supposed to be a regularization of the infinite continuum and the boundary conditions become irrelevant in the infinite volume limit. In the present case the lattice should be understood as a regularization of a finite volume flat torus and the periodic boundary condition is considered to be legitimate. In the continuum, however, neither parallel transporters, the continuum analog of lattice link variables, nor wave functions can be periodic on tori in the presence of a net magnetic flux.

This apparent mismatch between the continuum and the lattice can be resolved by noting that the periodic link variables (2.2) become singular in the classical continuum limit $a \to 0$. The singularities can be removed by a gauge transformation

$$\Lambda_0(x) = \exp\left[-2\pi i\sum_{\mu < \nu} m_{\mu\nu}\left[\frac{x_\mu}{L}\right]\frac{x_\nu}{L}\right], \tag{2.7}$$

where $[c]$ stands for the integer part of c, i.e., $[c] = n$ if $n \leq c < n + 1$ for some integer n. The $\Lambda_0(x)$ removes the troublesome parts of (2.2) and yields

$$\begin{aligned}
U_\mu^{\mathrm{a}}(x) &= \Lambda_0(x)U_\mu(x)\Lambda_0^\dagger(x + \hat{\mu}) \\
&= \exp\left[-\frac{2\pi i}{L^2}\sum_{\nu < \mu} m_{\mu\nu}x_\nu + ib_\mu\right] \\
&= \exp\left[-i\sum_{\nu < \mu} F_{\mu\nu}x_\nu + ib_\mu\right].
\end{aligned} \tag{2.8}$$

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This link variable has a smooth classical continuum limit and (2.4) becomes apparent in this gauge. The cost we must pay is the simple periodicity of the link variable. The gauge transformation (2.7) is not periodic under the shift $x \to x + L\hat{\mu}$ but satisfies

$$\Lambda_0(x + L\hat{\mu}) = \Lambda_\mu^{\mathrm{a}}(x)\Lambda_0(x). \qquad \left(\Lambda_\mu^{\mathrm{a}}(x) = \exp\left[-iL\sum_{\nu>\mu} F_{\mu\nu}x_\nu\right]\right) \qquad (2.9)$$

It changes the periodic boundary conditions to twisted ones. The gauge transformed wave functions

$$\psi^{\mathrm{a}}(x) = \Lambda_0(x)\psi(x) \qquad (2.10)$$

are subject to

$$\psi^{\mathrm{a}}(x + L\hat{\mu}) = \Lambda_\mu^{\mathrm{a}}(x)\psi^{\mathrm{a}}(x). \qquad (2.11)$$

The link variable (2.8) corresponds to axial gauge in the continuum. We can also work in the symmetric gauge by carrying out a further gauge transformation by

$$\Lambda^{\mathrm{sa}}(x) = \exp\left[\frac{i}{2}\sum_{\mu<\nu} F_{\mu\nu}x_\mu x_\nu\right]. \qquad (2.12)$$

The link variable in the symmetric gauge is given by

$$U_\mu^{\mathrm{s}}(x) = \exp\left[-\frac{i}{2}\sum_\nu F_{\mu\nu}x_\nu + ib_\mu\right]. \qquad (2.13)$$

In this gauge the link variables and the wave functions satisfy the boundary conditions

$$U_\mu^{\mathrm{s}}(x + L\hat{\nu}) = \Lambda_\nu^{\mathrm{s}}(x)U_\mu^{\mathrm{s}}(x)\Lambda_\nu^{\mathrm{s}\dagger}(x + \hat{\mu}), \qquad \psi^{\mathrm{s}}(x + L\hat{\mu}) = \Lambda_\mu^{\mathrm{s}}(x)\psi^{\mathrm{s}}(x), \qquad (2.14)$$

where $\Lambda_\mu^{\mathrm{s}}(x)$ is defined by

$$\Lambda_\mu^{\mathrm{s}}(x) = \exp\left[-\frac{iL}{2}\sum_\nu F_{\mu\nu}x_\nu\right]. \qquad (2.15)$$

This gauge is suitable for analyzing magnetic translation symmetry in the next section.

The constant phases b_μ in the link variables do not affect the magnetic field $F_{\mu\nu}$. In the continuum we can remove them by a suitable choice of the coordinate origin if $\det F_{\mu\nu} \neq 0$. The spectrum of the Hamiltonian does not depend on them. This is not the case for $\det F_{\mu\nu} = 0$ as is observed in odd dimensions. In lattice gauge theory we can also remove b_μ from the link variables by a further twisting of the boundary conditions. The eigenvalues depend on b_μ even if $\det F_{\mu\nu} \neq 0$ is satisfied. In this section we have not carried out this twisting, leaving the constant term intact.

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3 Magnetic translation symmetry on the lattice

In the continuum the Hamiltonian of a charged particle in a uniform magnetic field is not invariant under arbitrary translations. This is due to the coordinate dependence of the gauge potential. A discrete subgroup of the translations known as the magnetic translation [1], however, survives. It is a suitable combination of translations and gauge transformations. We expect that a similar situation also occurs on the lattice. In this section we pursue the conditions for the magnetic translations in lattice gauge theory.

We work in the symmetric gauge (2.13) and consider a shift of the lattice coordinates $x \to x + \ell$, where ℓ is an arbitrary integer vector. It is easy to verify that the link variables $U_\mu^{\mathrm{s}}(x)$ and $U_\mu^{\mathrm{s}}(x+\ell)$ are related by

$$U_\mu^{\mathrm{s}}(x+\ell) = \Omega_\ell^{\mathrm{s}}(x) U_\mu^{\mathrm{s}}(x) \Omega_\ell^{\mathrm{s}\dagger}(x+\hat{\mu}), \tag{3.1}$$

where $\Omega_\ell^{\mathrm{s}}(x)$ is given by

$$\Omega_\ell^{\mathrm{s}}(x) \;=\; \exp\left[-\frac{i}{2}\sum_{\mu,\nu} F_{\mu\nu}\ell_\mu x_\nu\right]. \tag{3.2}$$

The relation (3.1) is analogous to a gauge transformation. In general we cannot regard $U_\mu^{\mathrm{s}}(x+\ell)$ as a link variable associated with the link $(x,\hat{\mu})$ since it does not satisfy the boundary conditions (2.14). However, eq. (3.1) suggests a transformation $T_\ell : \psi^{\mathrm{s}} \to \psi_\ell^{\mathrm{s}}$ defined by

$$T_\ell \;:\; \psi_\ell^{\mathrm{s}}(x) = \Omega_\ell^{\mathrm{s}\dagger}(x)\psi^{\mathrm{s}}(x+\ell). \tag{3.3}$$

If $\psi^{\mathrm{s}}(x)$ is an eigenvector of the Hamiltonian and $\psi_\ell^{\mathrm{s}}(x)$ satisfies the twisted boundary conditions (2.14), then $\psi_\ell^{\mathrm{s}}(x)$ is also an eigenvector belonging to the same eigenvalue with $\psi^{\mathrm{s}}(x)$. In other words the transformation (3.3) is a symmetry of the Hamiltonian. We call this as magnetic translation symmetry on the lattice.

The requirement that (3.3) satisfies the boundary conditions (2.14) leads to the consistency conditions for $\Omega_\ell^{\mathrm{s}}(x)$

$$\Lambda_\mu^{\mathrm{s}}(x+\ell)\Omega_\ell^{\mathrm{s}}(x) = \Omega_\ell^{\mathrm{s}}(x+L\hat{\mu})\Lambda_\mu^{\mathrm{s}}(x). \tag{3.4}$$

This can be seen by noting that a translation $x \to x + L\hat{\mu} + \ell$ can be achieved in two different ways $x \to x + L\hat{\mu} \to x + L\hat{\mu} + \ell$ and $x \to x + \ell \to x + \ell + L\hat{\mu}$. The conditions (3.4) can be stated in terms of $m_{\mu\nu}$ as

$$\sum_\nu m_{\mu\nu}\ell_\nu \equiv 0 \mod L. \tag{3.5}$$

One can also arrive at the same result by working in other gauges. In the axial gauge (2.8) magnetic translations are implemented by

$$\Omega_\ell^{\mathrm{a}}(x) \;=\; \exp\left[-i\sum_{\mu<\nu} F_{\mu\nu}\ell_\mu x_\nu\right]. \tag{3.6}$$

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We will use this later.

We now turn to analyzing the constraints (3.5). We can block-diagonalize $m_{\mu\nu}$ into 2×2 anti-symmetric integer matrices as in the continuum [9, 3, 10]

$$m_{\mu\nu} = \sum_{\rho,\sigma} \mathcal{L}_\mu{}^\rho \mathcal{L}_\nu{}^\sigma \nu_{\rho\sigma}, \tag{3.7}$$

where $\mathcal{L} = (\mathcal{L}_\mu{}^\nu)$ is an integer matrix with $\det \mathcal{L} = 1$ and $\nu = (\nu_{\mu\nu})$ takes the form

$$\nu = \begin{pmatrix} 0 & \nu_1 & & & & & & \\ -\nu_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \nu_m & & & \\ & & & -\nu_m & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}. \tag{3.8}$$

The set of integers ν_p $(p = 1, \cdots, m)$ with $2m$ being the rank of $(m_{\mu\nu})$ can be chosen so that ν_p divides ν_{p+1} for $p = 1, \cdots, m-1$.

We write $\mathcal{L}^{-1}$ in terms of d integer vectors M_p, N_p and K_l $(a = 1, \cdots, m, \ l = 2m + 1, \cdots, d)$ as

$$\mathcal{L}^{-1} = (M_1, N_1, \cdots, M_m, N_m, K_{2m+1}, \cdots, K_d)^T \tag{3.9}$$

We also introduce the dual integer vectors M^{*p}, N^{*p} and K^{*l} by

$$\mathcal{L} = (M^{*1}, N^{*1}, \cdots, M^{*m}, N^{*m}, K^{*2m+1}, \cdots, K^{*d}). \tag{3.10}$$

Then (3.7) can be written as

$$m_{\mu\nu} = \sum_{p=1}^{m} \nu_p (M_\mu^{*p} N_\nu^{*p} - N_\mu^{*p} M_\nu^{*p}) \tag{3.11}$$

The set of d integer vectors M, N and K also generates the original lattice. We can expand an arbitrary integer vector ℓ as

$$\ell = \sum_{p=1}^{m} (\ell_M^p M_p + \ell_N^p N_p) + \sum_{l=1}^{d-2m} \ell_K^l K_l, \tag{3.12}$$

where ℓ_M^p, ℓ_N^p and ℓ_K^l are all integers. The constraints (3.5) for the magnetic translations ℓ can be simplified as

$$\nu_p \ell_M^p \equiv \nu_p \ell_N^p \equiv 0 \mod L, \tag{3.13}$$

whereas ℓ_K^l can be arbitrary.

To solve the conditions (3.13) we introduce a set of integers r_p, $s^p > 0$ and n^p by

$$n^p r_p = \nu_p, \qquad r_p s^p = L \tag{3.14}$$

where n^p and s^p are mutually prime. Then $\ell_M^p = js^p$ and $\ell_N^p = ks^p$ satisfy (3.13) for arbitrary integers j, k. We thus find three basic vectors generating magnetic translations

$$\ell_p^{(M)} = s^p M_p, \quad \ell_p^{(N)} = s^p N_p, \quad \ell_l^{(K)} = K_l. \tag{3.15}$$

In the following sections we need another set of integers r_p', $s_p' > 0$ and n'^p defined by

$$n'^p s_p' = n^p, \qquad r_p' s_p' = r_p, \tag{3.16}$$

where n'^p and r_p' are mutually prime.

4 Magnetic translations on 2d lattice

To illustrate magnetic translations on the lattice we consider a two dimensional system with a magnetic field $F_{12} = 2\pi\nu/L^2$. This is already of a standard form (3.7) and $\mathcal{L}$ is taken to be the identity matrix. The integer vectors M_1, N_1 and their dual are given by

$$M_1 = M^{*1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad N_1 = N^{*1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As in (3.14), we have a unique set of integers $n = n^1$, $r = r_1$ and $s = s^1$ for a given L and $\nu = \nu_1$. We also define another set of integers $n' = n'^1$, $r' = r_1'$ and $s' = s'^1$ by (3.16). We thus find two independent basic magnetic translations $T_x \equiv T_{sM_1}$ and $T_y \equiv T_{sN_1}$ corresponding to $x \to x + sM_1$ and $x \to x + sN_1$, respectively. In the axial gauge these act on the wave functions by

$$T_x \psi(x,y) = e^{\frac{2i\pi n' y}{sr'}} \psi(x+s,y), \qquad T_y \psi(x,y) = \psi(x,y+s), \tag{4.1}$$

where use has been made of (3.6). We have suppressed the label "a" standing for the axial gauge. As can be verified from (4.1), these operator satisfy

$$T_x T_y = e^{-\frac{2i\pi n'}{r'}} T_y T_x. \tag{4.2}$$

This gives $T_x^{r'} T_y = T_y T_x^{r'}$. We can find simultaneous eigenvectors of H, $T_x^{r'}$ and T_y. The eigenvalues of $T_x^{r'}$ can be written as $e^{\frac{2i\pi q}{s'}}$ by an integer q ($0 \le q < s'$) since T_x satisfies $(T_x^{r'})^{s'} = T_x^r = 1$. Similarly, we can write the eigenvalue of T_y as $e^{\frac{2i\pi p}{r}}$ with an integer $0 \le p < r$. Let us denote a simultaneous eigenvectors by $\psi_{p,q}$, then we have

$$\begin{aligned} H\psi_{p,q}(x,y) &= \lambda_{p,q}\psi_{p,q}(x,y), \\ T_x^{r'}\psi_{p,q}(x,y) &= e^{\frac{2i\pi q}{s'}}\psi_{p,q}(x,y), \\ T_y\psi_{p,q}(x,y) &= e^{\frac{2i\pi p}{r}}\psi_{p,q}(x,y). \end{aligned} \tag{4.3}$$

The magnetic translation T_x^m ($m = 0, 1, \cdots, r'-1$) maps an eigenstate of T_y belonging to an eigenvalue $e^{\frac{2i\pi p}{r}}$ to other eigenstate with an eigenvalue $e^{\frac{2i\pi(p+mn)}{r}}$. In other words, it changes the label $p \mod r$ to $p + mn \mod r$. Since $p + r'n \equiv p \mod r$, there are r' degenerate eigenvectors belonging to the eigenvalue $\lambda_{p,q}$. We thus find that the eigenvalue $\lambda_{p,q}$ depends only on $p \mod s'$ with respect to p.

The second and the third of (4.3) together with (4.1) imply that the wave functions at (x, y) and $(x + sr', y)$ or $(x, y + s)$ are related by

$$\psi_{p,q}(x + sr', y) = e^{-\frac{2i\pi n' y}{s} + \frac{2i\pi q}{s'}}\psi_{p,q}(x, y), \qquad \psi_{p,q}(x, y + s) = e^{\frac{2i\pi p}{r}}\psi_{p,q}(x, y). \qquad (4.4)$$

Noting the twisted periodicity in y, we can expand $\psi_{p,q}(x, y)$ in Fourier series as

$$\psi_{p,q}(x, y) = \frac{1}{\sqrt{s}}\sum_{j=0}^{s-1}\varphi_{p,q;j}(x)e^{\frac{2i\pi(p+jn'r)y}{L}}, \qquad (4.5)$$

where $\varphi_{p,q;j}(x)$ must satisfy

$$\varphi_{p,q;j+s}(x) = \varphi_{p,q;j}(x), \qquad \varphi_{p,q;j}(x + sr') = e^{\frac{2i\pi q}{s'}}\varphi_{p,q;j+1}(x). \qquad (4.6)$$

We see that the original L^2 components of $\psi_{p,q}(x, y)$ $(0 \leq x, y < L)$ can be expressed in terms $s^2 r'$ values of a one dimensional wave function $\varphi_{p,q}(x) \equiv \varphi_{p,q;0}(x)$ $(0 \leq x < s^2 r')$. It satisfies the twisted periodicity

$$\varphi_{p,q}(x + s^2 r') = e^{\frac{2i\pi sq}{s'}}\varphi_{p,q}(x) \qquad (4.7)$$

as one can see from (4.6). The magnetic translations put no further restrictions on $\varphi_{p,q}(x)$.

Our arguments so far does not depend on the detailed form of the Hamiltonian H. We can block-diagonalize it into r' matrices whatever form it is. This leads us to the Hamiltonian $\mathcal{H}_{p;j}$ defined by

$$H\psi_{p,q}(x, y) = \frac{1}{\sqrt{s}}\sum_{j=0}^{s-1}e^{\frac{2i\pi(p+jn'r)y}{L}}\mathcal{H}_{p;j}\varphi_{p,q;j}(x). \qquad (4.8)$$

It acts on the one dimensional system of a size $s^2 r'$ satisfying the twisted boundary condition (4.7). Since the eigenvalues of $\mathcal{H}_{p;j}$ do not depend on j, it is enough to consider the case $j = 0$ as mentioned above. We thus arrive at the eigenvalue equation for $\varphi_{p,q}(x)$

$$\mathcal{H}_p\varphi_{p,q}(x) = \lambda_{p,q}\varphi_{p,q}(x), \qquad (4.9)$$

where $\mathcal{H}_p$ stands for $\mathcal{H}_{p;0}$.

For tight-binding Hamiltonian $\psi_{p,q}(x, y)$ satisfies

$$e^{ib_x}\psi_{p,q}(x + 1, y) + e^{-ib_x}\psi_{p,q}(x - 1, y)$$
$$+ e^{\frac{2i\pi n' x}{s^2 r'} + ib_y}\psi_{p,q}(x, y + 1) + e^{-\frac{2i\pi n' x}{s^2 r'} - ib_y}\psi_{p,q}(x, y - 1) = \lambda_{p,q}\psi_{p,q}(x, y). \quad (4.10)$$

The equations for $\varphi_{p,q}(x)$ can be found by inserting (4.5) into the expression. The constant phase factors $e^{\pm ib_x}$ can be eliminated if we introduce

$$\tilde{\varphi}_{p,q}(x) = e^{ib_x x}\varphi_{p,q}(x). \qquad (4.11)$$

We thus obtain

$$\tilde{\varphi}_{p,q}(x + 1) + \tilde{\varphi}_{p,q}(x - 1) + \left\{2\cos\left(\frac{2\pi n' x}{s^2 r'} + \frac{2\pi p}{sr} + b_y\right) - \lambda_{p,q}\right\}\tilde{\varphi}_{p,q}(x) = 0. \qquad (4.12)$$
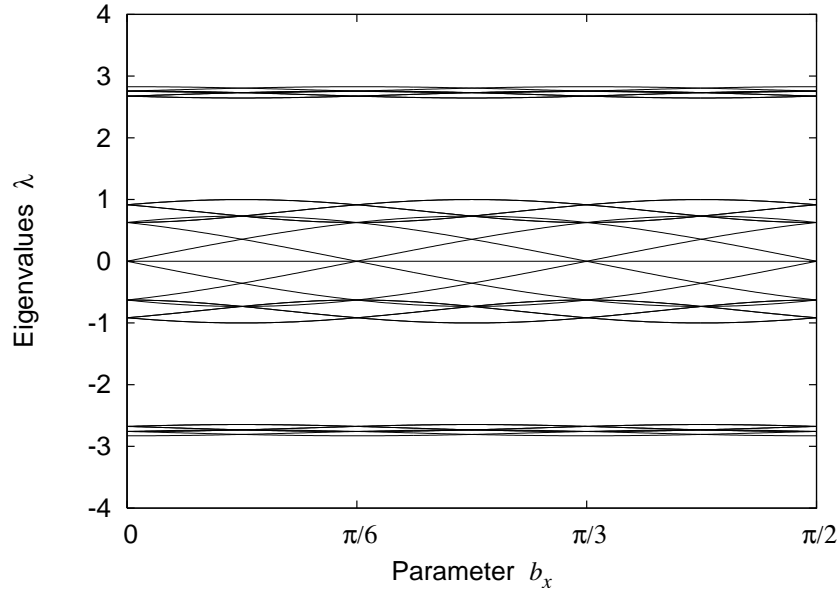
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Figure 1: Eigenvalues of tight-binding Hamiltonian in two dimensions are plotted for $0 \leq b_x \leq \pi/2$ and $b_y = 0$. We take $L = 6$, $\nu = 9$.

This is referred to as the Harper equation [11] in condensed matter physics. The periodicity (4.7) under the shift $x \to x + s^2 r'$ is further twisted by (4.11) and is given by

$$\tilde{\varphi}_{p,q}(x + s^2 r') = e^{\frac{2i\pi sq}{s'} + ib_x s^2 r'} \tilde{\varphi}_{p,q}(x). \tag{4.13}$$

The plot of the spectrum as a function of $\alpha = n'/s^2 r'$, i.e., magnetic flux per plaquette, is known as the butterfly diagram [2]. Unlike the continuum theories eigenvalues depends on the parameters b_x and b_y. The twisted periodicity (4.13) is invariant under the shift $b_x \to b_x + 2\pi/s^2 r'$. This implies that the eigenvalues are periodic in b_x and also in b_y by rotation symmetry with a period $2\pi/s^2 r'$. See Fig.1.

5   Extension to higher dimensions

We have shown that the magnetic translation symmetry can be used to constrain the wave functions on a two dimensional square lattice. We now extend this to higher dimensions.

Magnetic translation symmetry is apparent for special types of magnetic fields like (3.8) and the analysis for the two dimensional system in the previous section is applicable immediately. This is not the case for a general uniform magnetic field. Fortunately, it is always possible to transform $F_{\mu\nu}$ into the block-diagonal form (3.8) as mentioned in Sect. 3. In the continuum eigenvalue problems on magnetized tori can be solved by using this approach [12]. In this section we use the notation of Ref. [10].

We introduce oblique lattice coordinates ξ^p, η^p, χ^l $(p = 1, \cdots, m,\, l = 2m + 1, \cdots, d)$ by

$$x = \sum_p (M_p \xi^p + N_p \eta^p) + \sum_l K_l \chi^l, \tag{5.1}$$

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where M_p, N_p and K_l are oblique lattice vectors defined by (3.9). The oblique lattice coordinates are all integers and have one-to-one correspondence with x as can be seen from

$$\xi^p = \sum_\mu M_\mu^{*p} x_\mu, \qquad \eta^p = \sum_\mu N_\mu^{*p} x_\mu, \qquad \chi^l = \sum_\mu K_\mu^{*l} x_\mu. \tag{5.2}$$

In particular the unit translations $x \to x \pm \hat{\mu}$ correspond to the following shifts

$$\xi^p \to \xi^p \pm M_\mu^{*p}, \qquad \eta^p \to \eta^p \pm N_\mu^{*p}, \qquad \chi^l \to \chi^l \pm K_\mu^{*l}. \tag{5.3}$$

Conversely, the translation $\xi, \eta, \chi \to \xi \pm e_p, \eta, \chi$ can be realized by the shift $x \to x \pm M_p$, where e_p is the p-th unit vector with $(\xi + e_p)_q = \xi_q + \delta_{p,q}$. Similar things hold true for the unit translations in η or in χ.

The link variables in the symmetric gauge (2.13) can be written as

$$
\begin{aligned}
U_\mu^{\mathrm{s}}(x) &= \exp\left[-\frac{i\pi}{L^2} \sum_p \nu_p (M_\mu^{*p}\eta^p - N_\mu^{*p}\xi^p) + ib_\mu \right] \\
&= \exp\left[-i \sum_p \frac{\pi n'^p}{(s^p)^2 r'_p} (M_\mu^{*p}\eta^p - N_\mu^{*p}\xi^p) + ib_\mu \right],
\end{aligned}
\tag{5.4}
$$

where n'^p, r'_p and s^p are given by (3.14) and (3.16). To argue magnetic translations it is more convenient to work in axial gauge on the oblique lattice. This can be achieved by the following gauge transformation

$$
U_\mu^\alpha(x) = \Lambda^{\alpha\mathrm{s}}(x) U_\mu^{\mathrm{s}}(x) \Lambda^{\alpha\mathrm{s}\dagger}(x+\hat{\mu}) = \exp\left[i \sum_p \frac{2\pi n'^p}{(s^p)^2 r'_p} N_\mu^{*p}\left(\xi^p + \frac{1}{2} M_\mu^{*p} \right) + ib_\mu \right]
$$

$$
\text{with} \quad \Lambda^{\alpha\mathrm{s}}(x) = \exp\left[-i \sum_p \frac{\pi n'^p}{(s^p)^2 r'_p} \xi^p \eta^p \right]. \tag{5.5}
$$

In this gauge the twisted boundary conditions for the wave functions are given by

$$\psi(x + L\hat{\mu}) = \Lambda_\mu^\alpha(x)\psi(x)$$

$$\text{with} \quad \Lambda_\mu^\alpha(x) = \exp\left[-i \sum_p \frac{2\pi n^p}{s^p} M_\mu^{*p}\left(\eta^p + \frac{L}{2} N_\mu^{*p} \right) \right]. \tag{5.6}$$

We have suppressed the label α indicating the axial gauge for the wave function.

We can also find the periodicity of the wave function under shifts of the oblique lattice coordinates by L by a repeated use of (5.6) as

$$
\begin{aligned}
\psi(\xi + Le_p, \eta, \chi) &= e^{-\frac{2i\pi n^p}{s^p}\eta^p + i\pi\epsilon_p^{(M)}} \psi(\xi, \eta, \chi), \\
\psi(\xi, \eta + Le_p, \chi) &= e^{i\pi\epsilon_p^{(N)}} \psi(\xi, \eta, \xi), \\
\psi(\xi, \eta, \chi + Le_l) &= e^{i\pi\epsilon_l^{(K)}} \psi(\xi, \eta, \chi).
\end{aligned}
\tag{5.7}
$$

where ϵ's are integers defined by

$$\epsilon_p^{(M)} = \sum_{\mu<\nu} m_{\mu\nu} M_{p\mu} M_{p\nu}, \qquad \epsilon_p^{(N)} = \sum_{\mu<\nu} m_{\mu\nu} N_{p\mu} N_{p\nu}, \qquad \epsilon_l^{(K)} = \sum_{\mu<\nu} m_{\mu\nu} K_{l\mu} K_{l\nu}. \tag{5.8}$$

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We see that the wave function is either periodic or anti-periodic in η^p and χ^l.

We now turn to magnetic translation of the wave function. In the axial gauge (5.5) the link variables on the sites x and $x + \ell$ are related by

$$U_\mu^\alpha(x + \ell) = \Omega_\ell^\alpha(x) U_\mu^\alpha(x) \Omega_\ell^{\alpha\dagger}(x + \hat{\mu})$$

$$\text{with} \quad \Omega_\ell^\alpha(x) = \exp\left[-i \sum_p \frac{2\pi n'^p}{(s^p)^2 r_p'} \ell_M^p \eta^p\right],$$

where ℓ is given by (3.12). This leads to magnetic translation of the wave function

$$T_\ell \psi(\xi, \eta, \chi) = \Omega_\ell^{\alpha\dagger}(x) \psi(\xi + \ell_M, \eta + \ell_N, \chi + \ell_K). \tag{5.9}$$

Let us denote the generators of magnetic translations corresponding to the three basic translations (3.15) by $T_p^{(M)}$, $T_p^{(N)}$ and $T_l^{(K)}$, respectively. Then (5.9) yields

$$T_p^{(M)} \psi(\xi, \eta, \chi) = \exp\left[\frac{2i\pi n'^p}{s^p r_p'} \eta^p\right] \psi(\xi + s^p e_p, \eta, \chi),$$

$$T_p^{(N)} \psi(\xi, \eta, \chi) = \psi(\xi, \eta + s^p e_p, \chi), \tag{5.10}$$

$$T_l^{(K)} \psi(\xi, \eta, \chi) = \psi(\xi, \eta, \chi + e_l),$$

The generators of magnetic translations satisfy

$$T_p^{(M)} T_p^{(N)} = e^{-\frac{2i\pi n'^p}{r_p'}} T_p^{(N)} T_p^{(M)} \tag{5.11}$$

and all other combinations are commutative. They also satisfy

$$((T_p^{(M)})^{r_p'})^{s_p'} = e^{i\pi \epsilon_p^{(M)}}, \quad (T_p^{(N)})^{r_p} = e^{i\pi \epsilon_p^{(N)}}, \quad (T_l^{(K)})^L = e^{i\pi \epsilon_l^{(K)}}, \tag{5.12}$$

as can be seen from (5.7) and (5.10). These are a higher dimensional generalization of (4.1) and (4.2).

We may choose a set of commuting operators H, $(T_p^{(M)})^{r_p'}$, $T_p^{(N)}$, $T_l^{(K)}$ as in two dimensions and consider eigenvectors of these operators

$$H\psi_{k,p,q}(\xi, \eta, \chi) = \lambda_{k,p,q}\psi_{k,p,q}(\xi, \eta, \chi),$$

$$(T_p^{(M)})^{r_p'}\psi_{k,p,q}(\xi, \eta, \chi) = \exp\left[\frac{2i\pi}{s_p'}\left(q_p + \frac{1}{2}\epsilon_p^{(M)}\right)\right]\psi_{k,p,q}(\xi, \eta, \chi),$$

$$T_p^{(N)}\psi_{k,p,q}(\xi, \eta, \chi) = \exp\left[\frac{2i\pi}{r_p}\left(p_p + \frac{1}{2}\epsilon_p^{(N)}\right)\right]\psi_{k,p,q}(\xi, \eta, \chi),$$

$$T_l^{(K)}\psi_{k,p,q}(\xi, \eta, \chi) = \exp\left[\frac{2i\pi}{L}\left(k_l + \frac{1}{2}\epsilon_l^{(K)}\right)\right]\psi_{k,p,q}(\xi, \eta, \chi), \tag{5.13}$$

where $0 \le q_p < s_p'$, $0 \le p_p < r_p$ and $0 \le k_l < L$ are integers. These together with (5.10) implies

$$\psi_{k,p,q}(\xi + s^p r_p' e_p, \eta, \chi) = \exp\left[-\frac{2i\pi n'^p}{s^p}\eta^p + \frac{2i\pi}{s_p'}\left(q_p + \frac{1}{2}\epsilon_p^{(M)}\right)\right]\psi_{k,p,q}(\xi, \eta, \chi),$$

$$\psi_{k,p,q}(\xi, \eta + s^p e_p, \chi) = \exp\left[\frac{2i\pi}{r_p}\left(p_p + \frac{1}{2}\epsilon_p^{(N)}\right)\right]\psi_{k,p,q}(\xi, \eta, \chi), \tag{5.14}$$

$$\psi_{k,p,q}(\xi, \eta, \chi + e_l) = \exp\left[\frac{2i\pi}{L}\left(k_l + \frac{1}{2}\epsilon_l^{(K)}\right)\right]\psi_{k,p,q}(\xi, \eta, \chi).$$

The last two of these expressions enable us to write

$$\psi_{k,p,q}(\xi,\eta,\chi) = \frac{1}{\sqrt{s^1 \cdots s^m}} \sum_j \varphi_{k,p,q;j}(\xi) e^{\frac{2i\pi}{L}\left\{\sum_q \left(n'^q r_q j_q + p_q + \frac{1}{2}\epsilon_q^{(N)}\right)\eta^q + \sum_l \left(k_l + \frac{1}{2}\epsilon_l^{(K)}\right)\chi^l\right\}}, \quad (5.15)$$

where the sum is taken over integers $0 \leq j_p < s^p$ $(p = 1, \cdots, m)$. The Fourier coefficients $\varphi_{k,p,q;j}(\xi)$ must satisfy the following periodicity

$$\varphi_{k,p,q;j+s^p e_p}(\xi) = \varphi_{k,p,q;j}(\xi), \qquad \varphi_{k,p,q;j}(\xi + s^p r'_p e_p) = e^{\frac{2i\pi}{s'_p}\left(q_p + \frac{1}{2}\epsilon_p^{(M)}\right)} \varphi_{k,p,q;j+e_p}(\xi) \quad (5.16)$$

The first comes from the definition of $\varphi_{k,p,q;j}(\xi)$ and the second from the first of (5.14). We see that the original L^d components of the wave function $\psi_{k,p,q}(x)$ $(0 \leq x_\mu < L, \mu = 1, \cdots, d)$ can be obtained from the $(s^1)^2 r'_1 \cdots (s^m)^2 r'_m$ components of $\varphi_{k,p,q}(\xi) \equiv \varphi_{k,p,q;0}(\xi)$ $(0 \leq \xi^p < (s^p)^2 r'_p, p = 1, \cdots, m)$. It must satisfy the twisted boundary conditions

$$\varphi_{k,p,q}(\xi + (s^p)^2 r'_p e_p) = e^{\frac{2i\pi s^p}{s'_p}\left(q_p + \frac{1}{2}\epsilon_p^{(M)}\right)} \varphi_{k,p,q}(\xi) \quad (5.17)$$

and can be considered as a wave functions of some m dimensional system. The Hamiltonian $\mathcal{H}_{k,p;j}$ of the dimensionally reduced system is found by inserting (5.15) into the first of (5.13), i.e.,

$$H\psi_{k,p,q}(\xi,\eta,\chi) = \frac{1}{\sqrt{s^1 \cdots s^m}} \sum_j e^{\frac{2i\pi}{L}\left\{\sum_q \left(n'^q r_q j_q + p_q + \frac{1}{2}\epsilon_q^{(N)}\right)\eta^q + \sum_l \left(k_l + \frac{1}{2}\epsilon_l^{(K)}\right)\chi^l\right\}} \mathcal{H}_{k,p;j}\varphi_{k,p,q;j}(\xi).$$

$$(5.18)$$

It is enough to consider the case $j = 0$. We thus obtain

$$\mathcal{H}_{k,p}\varphi_{k,p,q}(\xi) = \lambda_{k,p,q}\varphi_{k,p,q}(\xi), \quad (5.19)$$

where we have defined $\mathcal{H}_{k,p} \equiv \mathcal{H}_{k,p;0}$. This together with (5.17) determines the wave functions $\varphi_{k,p,q}$, or equivalently $\varphi_{k,p,q;j}$, and the eigenvalues $\lambda_{k,p,q}$. They are defined on the m dimensional oblique lattice of a size $(s^1)^2 r'_1 \times \cdots \times (s^m)^2 r'_m$. We can then reconstruct $\psi_{k,p,q}$ by (5.15) and arrive at $r_1 \cdots r_m s'_1 \cdots s'_m L^{d-2m}$ wave functions forming a complete set.

For any fixed k and q there are $r_1 \cdots r_m$ wave functions $\psi_{k,p,q}$. They can be decomposed into $s'_1 \cdots s'_m$ sets of wave functions degenerated by the magnetic translation symmetry. For a given set of labels $p_q \mod s'_q$ $(q = 1, \cdots, m)$ there are exactly $r'_1 \cdots r'_m$ degenerate wave functions given by

$$(T_1^{(M)})^{h_1} \cdots (T_m^{(M)})^{h_m}\psi_{k,p,q}(\xi). \qquad (0 \leq h_q < r'_q, \ q = 1, \cdots, m) \quad (5.20)$$

We now apply the formalism developed so far to the tight-binding system described by

$$\sum_{\mu=1}^d \{U_\mu^\alpha(x)\psi_{k,p,q}(x+\hat\mu) + U_\mu^{\alpha\dagger}(x-\hat\mu)\psi_{k,p,q}(x-\hat\mu)\} = \lambda_{k,p,q}\psi_{k,p,q}(x), \quad (5.21)$$

where the link variables are given by (5.5). This can be converted to the form (5.19). To have a compact expression we introduce β's by the expansion

$$b_\mu = \sum_q (\beta_q^{(M)} M_\mu^{*q} + \beta_q^{(N)} N_\mu^{*q}) + \sum_l \beta_l^{(K)} K_\mu^{*l}, \quad (5.22)$$

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and define $\tilde{p}$, $\tilde{q}$ and $\tilde{k}$ by

$$\tilde{p}_p = p_p + \frac{1}{2}\epsilon_p^{(N)} + \frac{L}{2\pi}\beta_p^{(N)}, \quad \tilde{q}_p = q_p + \frac{1}{2}\epsilon_p^{(M)} + \frac{L}{2\pi}\beta_p^{(M)}, \quad \tilde{k}_l = k_l + \frac{1}{2}\epsilon_l^{(K)} + \frac{L}{2\pi}\beta_l^{(K)}. \tag{5.23}$$

Then (5.21) is reduced to

$$\sum_{\mu=1}^{d}\Big\{ e^{\frac{2i\pi}{L}\sum_q N_\mu^{*q}\{\frac{\nu_q}{L}(\xi^q+\frac{1}{2}M_\mu^{*q})+\tilde{p}_q\}+\frac{2i\pi}{L}\sum_l K_\mu^{*l}\tilde{k}_l+i\sum_q \beta_q^{(M)}M_\mu^{*q}}\varphi_{k,p,q}(\xi+M_\mu^*)$$

$$+e^{-\frac{2i\pi}{L}\sum_q N_\mu^{*q}\{\frac{\nu_q}{L}(\xi^q-\frac{1}{2}M_\mu^{*q})+\tilde{p}_q\}-\frac{2i\pi}{L}\sum_l K_\mu^{*l}\tilde{k}_l-i\sum_q \beta_q^{(M)}M_\mu^{*q}}\varphi_{k,p,q}(\xi-M_\mu^*)\Big\} = \lambda_{k,p,q}\varphi_{k,p,q}(\xi). \tag{5.24}$$

The phase factor containing $\beta^{(M)}$ can be removed by introducing $\tilde{\varphi}_{k,p,q}$ as

$$\tilde{\varphi}_{k,p,q}(\xi) = e^{i\sum_q \beta_q^{(M)}\xi^q}\varphi_{k,p,q}(\xi). \tag{5.25}$$

We finally obtain

$$\sum_{\mu=1}^{d}\Bigg\{ e^{\frac{2i\pi}{L}\sum_q N_\mu^{*q}\{\frac{\nu_q}{L}(\xi^q+\frac{1}{2}M_\mu^{*q})+\tilde{p}_q\}+\frac{2i\pi}{L}\sum_l K_\mu^{*l}\tilde{k}_l}\tilde{\varphi}_{k,p,q}(\xi+M_\mu^*)$$

$$+e^{-\frac{2i\pi}{L}\sum_q N_\mu^{*q}\{\frac{\nu_q}{L}(\xi^q-\frac{1}{2}M_\mu^{*q})+\tilde{p}_q\}-\frac{2i\pi}{L}\sum_l K_\mu^{*l}\tilde{k}_l}\tilde{\varphi}_{k,p,q}(\xi-M_\mu^*)\Bigg\} = \lambda_{k,p,q}\tilde{\varphi}_{k,p,q}(\xi),$$

$$\text{with} \quad \tilde{\varphi}_{k,p,q}(\xi+(s^p)^2 r'_p e_p) = e^{\frac{2i\pi s^p}{s'_p}\tilde{q}_p}\tilde{\varphi}_{k,p,q}(\xi) \tag{5.26}$$

This is a higher dimensional extension of the Harper equation (4.12). The twisted periodicity can be derived from (5.17) and (5.25). It is periodic under the shift $\beta^{(M)} \to \beta^{(M)} + 2\pi e_p/(s^p)^2 r'_p$. This implies that the spectrum is symmetric under the same shift as has been observed in two dimensions.

6 Summary and discussion

We have investigated magnetic translation symmetries on finite periodic lattices in arbitrary dimensions. We have shown that any lattice system with uniform background magnetic field possesses the symmetry. In continuum theories magnetic flux tensor $m_{\mu\nu}$ determines a unique magnetic translation group. In lattice theories, however, it depends on how the lattice size L is taken. This is due to the simple fact that only translations by a divisor of L is allowed on the lattice whereas ν_p is not necessary a divisor of L. The symmetry changes rapidly as one varies the magnetic field. Then the spectrum of H is very sensitive with respect to the applied magnetic field. These qualitative differences between the continuum and the lattice disappear in the continuum limit. We can always take the limit keeping any prescribed magnetic translation symmetry.

In the extreme case that there is no magnetic translation shorter than the period of the lattice, eigenvectors cannot be constrained by the magnetic translation symmetries. We cannot reduce the number of unknown variables in solving the eigenvalue problem. However,

it is always possible to dimensionally reduce the system. An immediate consequence of this is that any tight-binding system can be formulated by the generalized Harper equations. Dimensionally reduced systems may be more tractable than the original equations and be beneficial to further studies.

As an application of magnetic translation symmetry we can compute index of overlap Dirac operator for abelian gauge background. It is related to spectral asymmetry of hermitian Wilson-Dirac operator. Due to the topological invariance of the index it suffices to compute the spectral asymmetry for a uniform magnetic field belonging to the same topological sector. Since the degeneracy of each eigenvector is easily seen from the magnetic translation symmetry, we have only to compute the spectral asymmetry of the reduced Hamiltonian. This has been carried out in two dimensions. Extension to higher dimensions is certainly interesting. We will argue the magnetic translation symmetry in lattice fermion systems elsewhere.

References

[1] J. Zak, Phys. Rev. **134** (1964) A1602, A1607.

[2] D.R. Hofstadter, Phys. Rev. **B14** (1976) 2239.

[3] S. Tanimura, J. Math. Phys. **43** (2002) 5926 [hep-th/0205053];
M. Sakamoto and S. Tanimura, J. Math. Phys. **44** (2003) 5042 [arXive:`hep-th/0306006`].

[4] H. J. Schellnhuber and G. M. Obermair, Phys. Rev. Lett. **45** (1980) 276;
G. H. Wannier, J. Math. Phys. **21** (1980) 2844.

[5] L. Giusti, A. González-Arroyo, Ch. Hoelbling, H. Neuberger and C. Rebbi, Phys. Rev. **D65** (2002) 074506.

[6] T. Fujiwara, Prog. Theor. Phys. bf 107 (2002) 163 [arXive:`hep-lat/0012007`];
H. Kurokawa and T. Fujiwara, Phys. Rev. **D67** (2003) 025015 [arXive:`hep-lat/0206014`].

[7] M. Lüscher, Nucl. Phys. **B549** (1999) 295 [arXive:`hep-lat/9811032`].

[8] T. Fujiwara, H. Suzuki and K. Wu, Prog. Theor. Phys. **105** (2001) 789 [arXive:`hep-lat/0001029`].

[9] J. Igusa, "Theta Functions," Lemma 5, pp. 71-72, Springer-Verlag (Berlin) (1972).

[10] Y. Tenjinbayashi, H. Igarashi and T. Fujiwara, Annals Phys. **322** (2007) 460 [arXive:`hep-th/0506259`].

[11] P.G. Harper, Proc. Phys. Soc. Lond. **A68** (1955) 874.

[12] P. van Baal, Comm. Math. Phys. **94** (1984) 397.